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Theory of Complex Variables - MA 209
Problem Sheet - 2

1. Write the given complex number in polar form first using an argument $\theta \neq Arg(z)$ and then using $\theta = Arg(z)$.
   (a) 2
   (b) $-3i$
   (c) $5-5i$
   (d) $\frac{12}{\sqrt{3}+i}$
2. Use a calculator to write the given complex number in polar form first using an argument $\theta \neq Arg(z)$and then using $\theta = Arg(z)$
   (a) $-\sqrt{2}+\sqrt{7}i$
   (b) $-12-5i$
3. Find z_1z_2 and $\frac{z_1}{z_2}$.Write the number in the form of $a+ib$
   (a) $z_1 = \sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$
   (b) $z_2 = \sqrt{3}(\cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12}))$
4. Write each complex number in polar form.Finally write the polar form in the form $a+ib$
   (a) $(3-3i)(5+5\sqrt{3}i)$
   (b) $\frac{\sqrt{2}+\sqrt{6}i}{-1+\sqrt{3}i}$
5. Compute the indicated powers.
   (a) $(2-2i)^5$
   (b) $(\sqrt{3}(\cos(\frac{2\pi}{9}) + i\sin(\frac{2\pi}{9}))^6$
6. Write the complex number in polar form and then in the form of $a+ib$
   $\frac{[8(\cos(\frac{3\pi}{8})+i\sin(\frac{3\pi}{8}))]^3}{[2(\cos(\frac{\pi}{16})+i\sin(\frac{\pi}{16}))]^6}$
7. Use De Moivre's formula with $n=2$ to find trigonometric identities for $\cos 2\theta$ and $\sin 2\theta$
8. Use De Moivre's formula with $n=3$ to find trigonometric identities for $\cos 3\theta$ and $\sin 3\theta$
9. Find a positive integer n for which the equality holds.
   $(\frac{\sqrt{3}i}{2} + \frac{1}{2}i)^n = -1$
10. Suppose that $z = r(cos\theta + isin\theta)$. Describe geometrically the effect of multiplying z by a complex number of the form $z_1 = cos\alpha + isin\alpha$ when $\alpha > 0$ and when $\alpha < 0$.
11. Suppose $z = cos\theta + isin\theta$. If n is an integer, evaluate $z^n + \bar{z}^n$and $z^n - \bar{z}^n$.
12. Write an equation that relates $arg(z)$ to $arg(1/z)$, $z \neq 0$.
13. Are there any special cases in which $Arg(z_1z_2) = Arg(z_1) + Arg(z_2)$? Proveyour assertions.
14. How are the complex numbers z_1 and z_2 related if $arg(z_1) = arg(z_2)$?
15. Describe the set of points z in the complex plane that satisfy $arg(z) = \pi/4$.

16. Suppose z_1, z_2, and z_1z_2 are complex numbers in the first quadrant and that the points $z = 0, z = 1, z_1, z_2$, and z_1z_2 are labeled O, A, B, C, and D,respectively. Discuss how the triangles OAB and OCD are related.

17. Suppose $z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $z_2 = r_2(cos\theta_2 + isin\theta_2)$. If $z_1 = z_2$, then how are r_1 and r_2 related? How are θ_1 and θ_2 related?

18. Suppose z_1 is in the first quadrant. For each z_2, discuss the quadrant in which $z_1z_2$could be located.

    (a) $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

    (b) $z_2 = -1$

19. Suppose z_1, z_2, z_3, and z_4 are four distinct complex numbers. Interpret geometrically: $\arg(\frac{z_1-z_2}{z_1-z_2}) = \frac{\pi}{2}$

20. For the following problems compute all roots. Give the principal nth root in each case. Sketch the roots $w_0, w_1, ..., w_{n-1}$ on an appropriate circle centered at the origin.

    (a) $(8)^{\frac{1}{3}}$

    (b) $(-125)^{\frac{1}{3}}$

    (c) $(-1+i)^{\frac{1}{3}}$

    (d) $(-1-\sqrt{3}i)^{\frac{1}{4}}$

    (e) $(\frac{1+i}{\sqrt{3}+i})^{(1/6)}$

21. Use the fact that $8i = (2+2i)^2$ to find all solutions of the equation $z^2 - 8z + 16 = 8i$.

22. Show that the n nth roots of unity are given by
    $(1)^{(1/n)} = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})\ k = 0, 1, 2, ..., n-1$
    (a)Find n nth roots of unity for $n = 3, n = 4, n = 5$
    (b)Carefully plot the roots of unity found in part (b).Sketch the regular polygons formed with the roots as vertices.

23. Suppose ω is a cube root of unity corresponding to k = 1 in the last problem.
    (a) How are ω and ω^2 related?
    (b)Verify by direct computation that $1 + \omega + \omega^2 = 0$.
    (c) Explain how the result in part (b) follows from the basic definition that ω is a cube root of 1, that is, $\omega^3 = 1$. [Hint: Factor]

24. For a fixed n, if we take k = 1 in Problem 22 , we obtain the root $\omega_n = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$
    Explain why the n nth roots of unity can then be written $1, \omega_n, \omega_n^2, ..., \omega_n^{n-1}$

25. Consider the equation $(z+2)^n + z^n = 0$, where n is a positive integer. By any means, solve the equation for z when n = 1. When n = 2.

26. Consider the equation in Problem 25.
    (a) In the complex plane, determine the location of all solutions z when n = 5. [Hint: Write the equation in the form $[(z+2)/(-z)]^5 = 1$ and use part (a) of Problem 22.]
    (b) Reexamine the solutions of the equation in Problem 25 for n = 1 and n = 2.

27. Let n be a fixed natural number.Put $\omega_n = cis(\frac{2\pi}{n})$. Show that $1 + \omega_n + \omega_n^2 + \omega_n^3 + \ + \omega_n^{n-1} = 0$. [Hint: Multiply the sum $1 + \omega_n + \omega_n^2 + \omega_n^3 + \ + \omega_n^{n-1}$ by $\omega_n - 1$.]

28. Suppose n denotes a nonnegative integer. Determine the values of n such that $z^n = 1$ possesses only real solutions. Defend your answer with sound mathematics.

29. Discuss: A real number can have a complex nth root. Can a nonreal complex number have a real nth root?

30. Suppose w is located in the first quadrant and is a cube root of a complex number z. Can there exist a second cube root of z located in the first quadrant? Defend your answer with sound mathematics.

31. Suppose z is a complex number that possesses a fourth root w that is neither real nor pure imaginary. Explain why the remaining fourth roots are neither real nor pure imaginary.

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